Extra Credit 6

A king decided to check how wise are his wisemen. The next day each of them will be placed in a single cell, no communication will be between them. The king's guard will in an arbitrary order bring a wiseman in a separate room with the switch (which is originally off). A wiseman can see the switch and decide whether to switch it or keep it in the same position. Wisemen (and you) need to find a strategy that will guarantee at some point for one of the wisemen to claim that all of his fellow wisemen visited the switch room at least once.

Wiseman Rafid counts the times he finds the light on and ensures that it is always off when he leaves the room. Everyone else turns on the light the first time they find it off and never touches it again. This way, between visits of Wiseman Rafid, at most one Wiseman, will turn on the light, and no Wiseman turns it on more than once. Therefore, the number of times Wiseman Rafid finds the light on is no more than the number of different Wisemen that have entered the room. Each prisoner knows he has been counted once he has turned the light on since he is the only one who touched the switch since Wiseman Rafid last visited. When Wiseman Rafid counts to n-1, he knows everyone has visited the room.

What does optimal mean here? It could only reasonably imply that the Wisemen are freed in the shortest time. So, what is the expected time they must wait until Wiseman Rafid has counted to n-1? This is a rather elaborate calculation in probability, so the Wisemen turn to the actuary.

He explains that using Bayes theorem,

P(X|Y) · P(Y) = P(X&Y) = P(Y|X) · P(X)

and the linearity of expected value,

E(X|Y) · P(Y) + E(X|~Y) · P(~Y) = E(X)

you can calculate the expected time in the dungeon-like this:

Suppose Wiseman Rafid has just visited the room, and let K be the number of days that pass before his next visit (so he visits again K+1 days from now), let n be the number of Wisemen, let c be the number of times he has found the light on so far, and let P(ON) and P(OFF) be the probabilities that he finds the light on or off on his next visit. Then E(K) = n - 1, P(K = k) = 1/n·((n-1)/n)k, P(K = k & OFF) = 1/n·(c/n)k, which are fairly obvious.

Summing the last formula over all k gives P(OFF) = 1/(n-c). Bayes theorem then gives P(K = k|OFF) = (1-c/n)·(c/n)k, and from this you can calculate E(K|OFF) = c/(n-c) and linearity gives

E(K|ON) = ((n-1)(n-c)-c/(n-c))/(n-c-1).

Now let m be the number of times Wiseman Rafid visits and L be the number of days that pass before he next finds it on. Each time he finds it is off, c does not change, so all the calculations regarding the time until his next visit also do not change.

Therefore, the expected number of days until he next finds the light on is found by summing over all possible m to get the expected total time wasted on visits where the light is off, plus the expected time for the one visit where it was on. This gives

E(L) = (1+E(K|ON))P(ON) + sum(m(1+E(K|OFF))P(OFF)m

= n(1/(n-c-1) - 1/(n-c) + 1 - 1/(n-c)2).

Now we know how long we expect to wait from count = c to count = c+1. Therefore, we must sum this up from c = 0 to c = n-2 to find the total expected time E(T). The result is E(T) = n2 - n/(n-1) - a, where a = S(1/c2) from 2 to n. Putting n = 100 into this gives 9935.5 days, which is 26.2 years.

But (continues the actuary) this is absurdly long to wait. Simple probability shows that we can be almost certainly much sooner than this. The probability that on day d, the count is c is P(c, d), which is obviously equal to P(c-1,d-1)·(1-(c-1)/n) + P(c, d-1)·(c/n). Of course, P(0, 0) = P(1, 1) = 1 and P(1,0) = 0, so we can recursively calculate the probability P(n, d). It turns out that P(100,1146) = 0.999, and P(100,1375) = 0.9999, P(100,1604) = 0.99999, and P(1833) = 0.999999. That means that in 3.14 years, we have a less than 1/1000 chance of failing, and in exactly five years and a week, we have less than one in a million chances of failing. I say we should wait five years and then say, "Let us out; we've all seen the light!"

**The dumbed down solution would be the first wiseman selected is in charge of turning the light on whenever it is found off. Each other Wiseman is to turn the light off the very first time they find it on, otherwise they are to leave it in the state they found it. When the first Wiseman selected turns the light on for the nth time, they can safely assume that all Wisemen have been in the interrogation room.**